Kelvin Model Equivalent to a Body with Viscoelastic Behaviour

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In the paper, I have brought in a method whereby a body with rheological behaviour is associated to a dynamic-equivalent Kelvin model. The equalization is made so that the forced vibration caused for Kelvin model by a harmonic external impulse is identical to the forced vibration caused in the case of the studied rheological model. It was experimentally determined the elasticity modulus and damping factor for samples with Dammar bio-matrix, reinforced with silk, flax or hemp fabric. Modifying samples dimensions, It was studied the vibrations frequency effect on the studied mechanical properties.

Keywords: Kelvin model, elasticity modulus, damping factor

Under the action of external forces, any body from nature has different deformations. The property of a body of returning to its initial shape, when external forces stop acting, is called elasticity. In the case in which the body does not return to its initial shape, it is said to have an elasto-plastic behaviour. The deformations are kept constant to an elastic body or to an elasto-plastic body with constant loads.

In reality, the bodies from nature have, along the mentioned elastic and plastic properties, the viscosity property that demonstrates that under constant external loads, the body deformations vary in time, phenomenon known as creep. Also, forcing the body to remain into constant deformation phase, the stresses inside the body vary in time, phenomenon known as relaxation.

In papers [1-3] there were implemented fading study methods for two rheological models, a generalised Maxwell body and a generalised Zener body. Also in these papers there were developed two sets of mathematical models. In [4] and subsequently [5] both models were revised and shown to be equivalent.

The rheological models that characterize fluids of non-newtonian drilling were studied in papers [8-9].

Rheological models based on generalized Maxwell and Zener bodies were used in studying viscoelastic polymers behaviour in [10-12]. These generalized rheological models are characterized by relaxation strokes or frequencies that have a reciprocity relation. There were analyzed the equivalent relations between these generalized rheological models, and the complex modulus along with relaxations functions were determined from schematic diagrams analysis.

The damping study on rheological models with viscoelastic behaviour is made in many papers. In [13] and [14] there are presented damped vibrations analysis methods for straight bars with different conditions at the ends. These methods are used in [15] and [16] for determining the damping on sandwich bars with polypropylene honeycomb core, and in [17] for composite bars reinforces with carbon and Kevlar.

The materials rheological behaviour is proportionally described by relations that connect stresses not only with specific strains, but with their time derivatives. We will call a linear model a rheological body, where between the stress Laplace transform (noted with $\overline{\sigma}$) and specific strain Laplace transform (noted with $\overline{\varepsilon}$) there is a linear relation as:

$$\overline{\sigma}(s) = E(s)\cdot \overline{\varepsilon}(s), \tag{1}$$

where E(s) is the model complex characteristic.

In table 1 there are presented the main linear models.

The fundamental elements are Hooke and Newton models, through their serial and in-line connection there are obtained other linear models. Generally, the complex characteristic E(s) is:

$$E(s) = \frac{Q(s)}{R(s)}, \tag{2}$$

where Q(s) and R(s) are polynoms in s, Q(s) polynom degree being equal or bigger with a unit than R(s) polynom degree.

The mathematical model of straight bar vibrations, neglecting rotational inertia of bar section, is:

$$m\ddot{w} - \frac{\partial^2}{\partial x^2}\iint_{(S)} y\,\sigma\, dS = q, \tag{3}$$

where:

- w(x, t)is the transversal displacement of bar medial fibre;
- m is the mass per bar unit length;
-q(x, t) is the external force distribuited on bar unit length.

Implementing in relation (3) the Laplace time transform and taking into account the relation (1), we obtain:

$$m\left(s^2\overline{w}(x,s) - s\,f(x) - g(x)\right) - \frac{\partial^2}{\partial x^2}\iint_{(S)} y\, E(s)\overline{\varepsilon}\, dS = \overline{q}(x,s), \tag{4}$$

where $f(x) = w(x,0)$ şi $g(x) = \dot{w}(x,0)$

In case of thin bars, Bernoulli's hypothesis is considered valid, according to which any bar section on all vibration period is normal to mean fibre. With this hypothesis, we have

$$\varepsilon = -y\frac{\partial^2 w}{\partial x^2}. \tag{5}$$

It is obtained the bar vibration equation in Laplace images:

$$m\left(s^2\overline{w} - s\,f(x) - g(x)\right) + E(s)I\frac{\partial^4 w}{\partial x^4} = \overline{q}, \tag{6}$$

where $I = \iint_{(S)} y^2 dS$ is the geometrical rotative moment of the bar section.

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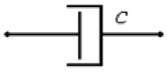
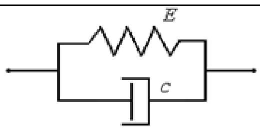
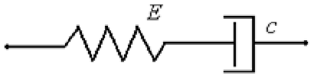
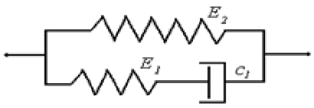
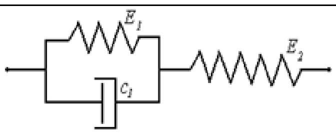
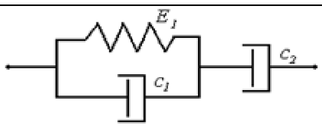
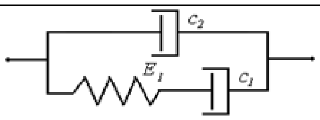
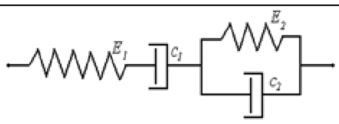
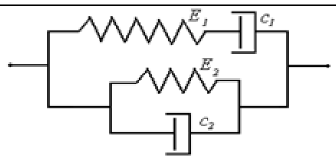
Model labeling	Typification	Complex characteristic
Hooke		$E(s)=E$
Newton		$E(s)=c\,s$
Kelvin		$E(s)=c\,s+E$
Maxwell		$E(s)=\dfrac{E\,c\,s}{c\,s+E}$
Zener-1		$E(s)=\dfrac{c_1(E_1+E_2)s+E_1E_2}{c_1\,s+E_1}$
Zener-2		$E(s)=\dfrac{E_2(c_1\,s+E_1)}{c_1\,s+E_1+E_2}$
Lethersich-1 (standard fluid)		$E(s)=\dfrac{c_1c_2\,s^2+E_1c_2\,s}{(c_1+c_2)s+E_1}$
Lethersich-2 (standard fluid)		$E(s)=\dfrac{c_1\,c_2\,s^2+E_1(c_1+c_2)s}{c_1\,s+E_1}$
Burges		$E(s)=\dfrac{E_1\,c_1\,s(c_2\,s+E_2)}{E_1\,c_1\,s+(c_1\,s+E_1)(c_2\,s+E_2)}$
Standard		$E(s)=\dfrac{E_1\,c_1\,s+(c_1\,s+E_1)(c_2\,s+E_2)}{c_1\,s+E_1}$

Table 1

We will consider the initial conditions null, and the bar is loaded with a harmonic external force with pulsation p:

$$q(x,t)=q_0(x)\sin\ pt. \tag{7}$$

Equation (6) becomes:

$$m\,s^2\,\overline{w}+E(s)\,I\,\frac{\partial^4 w}{\partial x^4}=\frac{q_0(x)\,p}{s^2+p^2}. \tag{8}$$

We consider that the bar vibration has the form:

$$w(x,t)=w(0,t)S(\lambda\,x)+\frac{1}{\lambda}w'(0,t)T(\lambda\,x)+ \\ +\frac{1}{\lambda^2}w''(0,t)U(\lambda\,x)+\frac{1}{\lambda^3}w'''(0,t)V(\lambda\,x), \tag{9}$$

where S(λ x), T(λ x), U(λ x), V(λ x) are Krylov functions given by the relations:

$$S(\lambda\,x)=\frac{1}{2}(ch\,\lambda\,x+\cos\lambda\,x),\ T(\lambda\,x)=\frac{1}{2}(sh\,\lambda\,x+\sin\ \lambda\,x), \\ U(\lambda\,x)=\frac{1}{2}(ch\,\lambda\,x-\cos\lambda\,x),\ V(\lambda\,x)=\frac{1}{2}(sh\,\lambda\,x-\sin\ \lambda\,x). \tag{10}$$

The solution (8) has certain advantages. One of these is the fact that the functions w(0, t), w'(0, t), w''(0, t) and w'''(0, t) can be explained as transversal section displacement and rotation of the deformed bar, a measure proportional to beding moment and a measure proportional to shearing force in transversal section x = 0. Two of these measures are deleted for random limit conditions.

From limit conditions in the other side of bar, there are determined the values series the constant λ has.

The solution (9) verifies the condition:

$$\frac{\partial^4 w(x,t)}{\partial x^4}=\lambda^4\,w(x,t). \tag{11}$$

From (8), it immediately results that:

$$\overline{w}(x,s)=\frac{p\,q_0(x)}{(m\,s^2+\lambda^4\,I\,E(s))(s^2+p^2)}. \tag{12}$$

Laplace inverting the relation (12), we obtain:

$$w(x,t)=w_a(x,t)+A(x)\cos pt+B(x)\sin\ pt, \tag{13}$$

where $w_a(x, t)$ is a vibration component that is attenuated in time, A(x) and B(x) and are given by the relations

$$A(x)=\frac{q_0(x)}{2i}\left[\frac{1}{\lambda^4\,I\,E(i\,p)-m\,p^2}-\frac{1}{\lambda^4\,I\,E(-i\,p)-m\,p^2}\right], \tag{14}$$

$$B(x)=\frac{q_0(x)}{2}\left[\frac{1}{\lambda^4\,I\,E(i\,p)-m\,p^2}+\frac{1}{\lambda^4\,I\,E(-i\,p)-m\,p^2}\right]. \tag{15}$$

If the component $w_a(x, t)$ is attenuated in time, harmonic components with pulsation p, equal to the external force one, are kept as long as the external harmonic load exists.

In the case of Kelvin model, the amplitudes given by (14) and (15) have the next form:

$$A(x) = \frac{-q_0(x)\lambda^4 I c p}{\left(\lambda^4 I E - m p^2\right)^2 + \left(\lambda^4 I c p\right)^2}, \tag{16}$$

$$B(x) = \frac{q_0(x)\left(\lambda^4 I E - m p^2\right)}{\left(\lambda^4 I E - m p^2\right)^2 + \left(\lambda^4 I c p\right)^2}. \tag{17}$$

Finding Kelvin model equivalent to a linear rheological model means finding elasticity modulus E and damping factor c of Kelvin model for which the forced vibration produced by an external excitation of type (7) is identical with the forced vibration obtained from the studied rheological model.

This identity is made through the given amplitudes equalization from relations (14) and (15) with the amplitudes obtained from relations (16) and (17). In this way, it is attained:

$$E(p) = \frac{1}{2}\left(E(i p) + E(-i p)\right), \tag{18}$$

$$c(p) = \frac{1}{2 p i}\left(E(i p) - E(-i p)\right). \tag{19}$$

These characteristics values for the specified main rheological models are:

- for Maxwell model

$$E(p) = \frac{E c^2 p^2}{E^2 + c^2 p^2}; \quad c(p) = \frac{E^2 c}{E^2 + c^2 p^2};$$

- for Zener-1 model

$$E(p) = \frac{E_1^2 E_2 + p^2 c_1^2 (E_1 + E_2)}{E_1^2 + p^2 c_1^2}; \quad c(p) = \frac{E_1^2 c_1}{E_1^2 + p^2 c_1^2};$$

- for Zener-2 model

$$E(p) = \frac{E_1 E_2 (E_1 + E_2) + p^2 c_1^2 E_2}{(E_1 + E_2)^2 + p^2 c_1^2}; \quad c(p) = \frac{c_1 E_2^2}{(E_1 + E_2)^2 + p^2 c_1^2};$$

- for Lethersich-1 model

$$E(p) = \frac{p^2 E_1 c_2^2}{E_1^2 + p^2 (c_1 + c_2)^2}; \quad c(p) = \frac{E_1^2 c_2 + c_1 c_2 p^2 (c_1 + c_2)}{E_1^2 + p^2 (c_1 + c_2)^2};$$

- for Lethersich-2 model

$$E(p) = \frac{p^2 E_1 c_1^2}{E_1^2 + p^2 c_1^2}; \quad c(p) = \frac{p^2 c_1^2 c_2 + E_1^2 (c_1 + c_2)}{E_1^2 + p^2 c_1^2};$$

- for Burgers model

$$E(p) = \frac{p^2 E_1 c_1^2 \left(p^2 c_2^2 + E_1 E_2 + E_2^2\right)}{\left(E_1 E_2 - p^2 c_1 c_2\right)^2 + p^2 \left(E_1 c_1 + E_2 c_1 + E_1 c_2\right)^2};$$

$$c(p) = \frac{E_1^2 E_2^2 c_1 + p^2 E_1^2 c_1 c_2 (c_1 + c_2)}{\left(E_1 E_2 - p^2 c_1 c_2\right)^2 + p^2 \left(E_1 c_1 + E_2 c_1 + E_1 c_2\right)^2};$$

- for standard model

$$E(p) = \frac{E_1^2 E_2 + p^2 c_1^2 (E_1 + E_2)}{E_1^2 + p^2 c_1^2}; \quad c(p) = \frac{p^2 c_1^2 c_2 + E_1^2 (c_1 + c_2)}{E_1^2 + p^2 c_1^2}.$$

Experimental part

I have made three sets of samples, depending on reinforcement, using as matrix a Dammar bio-resin (precisely, Dammar 60 % natural resin and 40% epoxy resin). Each set contained samples of 5 mm thickness and 10 mm,15 mm, 20 mm and 25 mm widths.

The reinforcement for the three sets of samples was:

- blending fabric of 60 % silk and 40 % cotton (hereinafter represented as silk), with specific mass of ; I have used 20 layers, the obtained composite having the resin mass ratio of 0.51, the resin volume ratio of 0.61 and density of 1.23 g/cm^3;

- blending fabric of 40 % cotton and 60 % flax (hereinafter represented as flax), with specific mass of 240 g/m^2; I have used 12 layers, the obtained composite having the resin mass ratio of 0.52, the resin volume ratio of 0.57 and density of 1.1 g/cm^3;

- hemp fabric, with specific mass of 352 g/m^2; I have used 6 layers, the obtained composite having the resin mass ratio of 0.62, the resin volume ratio of 0.66 and density of 1.15 g/cm^3.

In figure 1, there are presented the samples with reinforcement of hemp fabric.

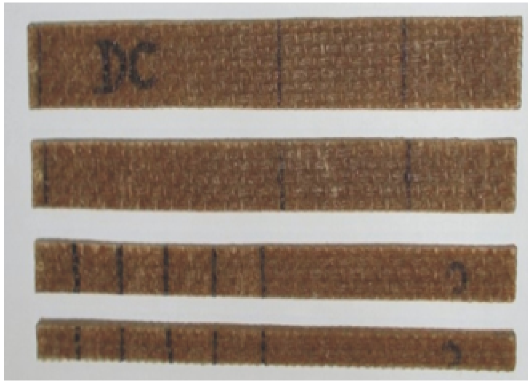


Fig. 1. Composite material samples reinforced with hemp fabric

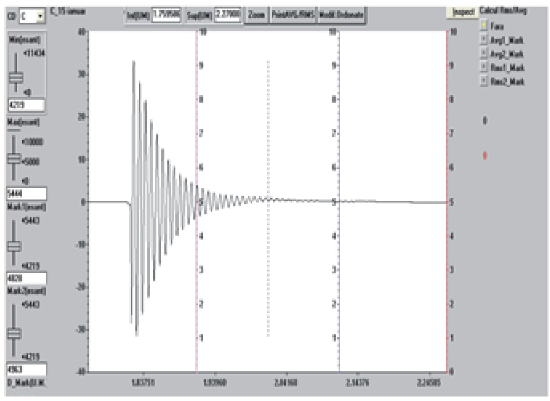


Fig. 2. Vibration experimental recording for sample reinforced with hemp, of 10 mm width

I have experimentally determined the damping factor for these sets of samples. The studied samples were embedded to an end, the other free one being attached to a 20 g additional mass, in order to modify the oscillation frequency. The free length for each studied bar was of 100 mm.

The used measuring devices were:

- data acquisition system SPIDER 8, connected through USB port to a notebook;

-data acquisition set was made through CATMAN EASY software, that connected the two entities;

- signal conditioner NEXUS 2692-A-0I4 connected to the system SPIDER 8;

- accelerometer with sensitivity of 0.04 pC/ms^{-2} connected to the signal conditioner.

The measuring domain for frequency was set from 0 - 2.400 Hz from SPIDER 8. To eliminate the errors entered by the experimental system, we have made for each measurement a Butterworth *High Pass* type filtering of 3 Hz frequency.

In figure 2, the vibration experimental recording for sample reinforced with hemp, of 10 mm width, is presented.

In figure 3, the damping factor determination for figure 2 recording is shown.

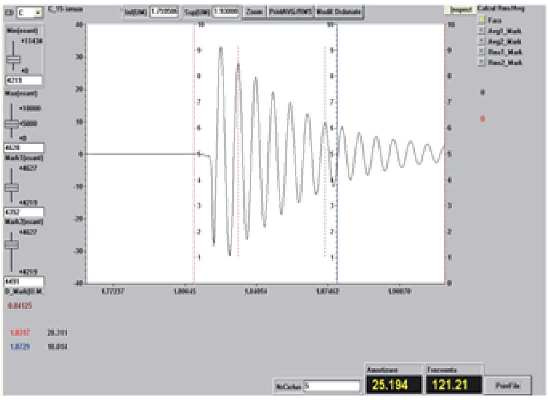


Fig. 3. The damping factor determination for figure 2 recording

Likewise, all experimental recordings were processed. From these recordings, through the methods presented in [16] and [17], we have determined the damping factor and the elasticity modulus.

In figures 4-6, there are presented the damping factor and elasticity modulus variations depending on oscillation frequency, for the three types of studied materials.

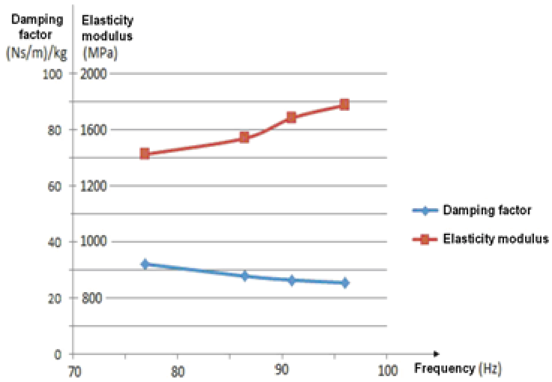


Fig. 4. Damping factor and elasticity modulus variation depending on frequency, for samples reinforced with silk

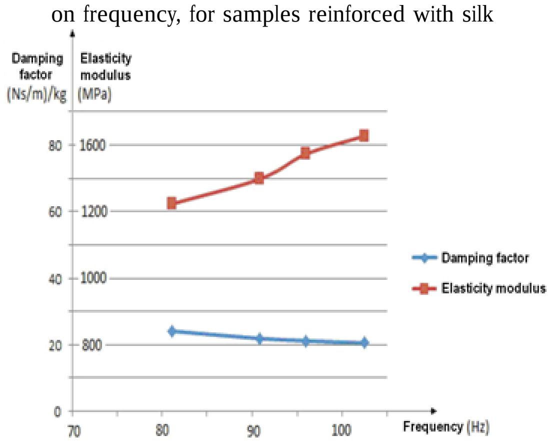


Fig. 5. Damping factor and elasticity modulus variation depending on frequency, for samples reinforced with flax

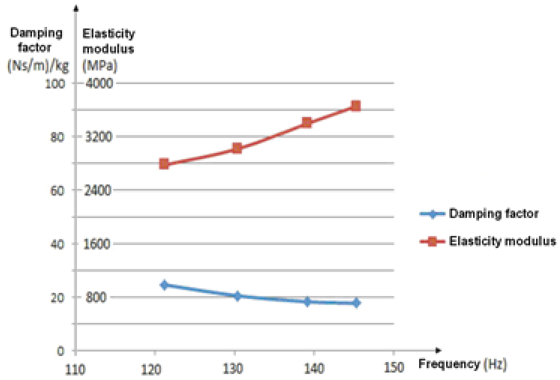


Fig. 6. Damping factor and elasticity modulus variation depending on frequency, for samples reinforced with hemp

Conclusions

Based on the presented data (both theoretical and experimental ones), it can be concluded that any linear rheological model can be assimilated with a Kelvin model, but whose characteristics depend on external force pulsation.

In figures 7 and 8, there are presented the theoretical variations depending on oscillation frequency, for elasticity modulus and damping factor.

It is observed that the equivalent Kelvin model elasticity modulus increases with external force pulsation, conversely the damping factor decreases with external pulsation increase.

In figures 7 and 8 we used the notations:

$$E_\infty = \lim_{p\to\infty} E(p); \quad E_0 = \lim_{\substack{p\to 0\\ p>0}} E(p); \qquad (20)$$

$$c_\infty = \lim_{p\to\infty} c(p); \quad c_0 = \lim_{\substack{p\to 0\\ p>0}} c(p). \qquad (21)$$

In table 2, these limit values for the shown main rheological models are presented.

Damping factors values analysis shows that these coefficients must be experimentally determined for each material and sample type, being hard to conclude a

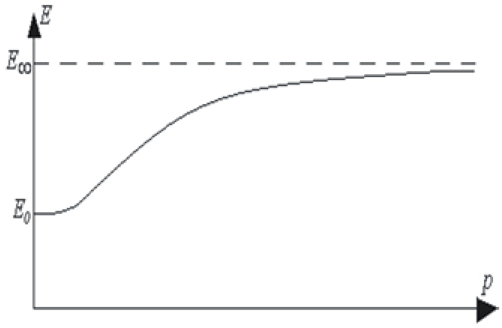


Fig. 7. Elasticity modulus variation

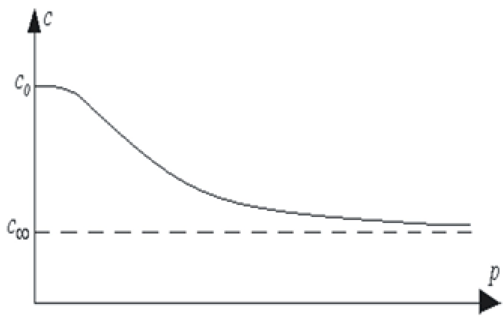


Fig. 8. Damping factor variation

Model labeling	E_0	E_∞	c_0	c_∞
Maxwell	0	E	c	0
Zener-1	E_2	E_1+E_2	c_1	0
Zener-2	$\frac{E_1 E_2}{E_1+E_2}$	E_2	$\frac{c_1 E_2^2}{(E_1+E_2)^2}$	0
Lethersich-1	0	$\frac{E_1 c_2^2}{(c_1+c_2)^2}$	c_2	$\frac{c_1 c_2}{c_1+c_2}$
Lethersich-2	0	E_1	c_1+c_2	c_2
Burgers	0	E_1	c_1	0
Standard	E_2	E_1+E_2	c_1+c_2	c_2

Table 2

quantitative connection with parameters that directly or indirectly influence the damping.

The damping factors values can depend on several factors as: the sample dimensions, the specific mass or the material quantity from the sample, elastical and damping properties of component materials.

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